How Can Governments Borrow so Much?

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Abstract

We analyze the determinants of government debt under the twin assumptions that governments have limited horizons and default only when government income falls short of debt service requirements. We derive a government's maximum sustainable debt ratio, that is, the debt ratio chosen by a myopic government whose horizon does not extend beyond its current term in office. Maximum sustainable debt varies across countries, consistent with Reinhart, Rogoff, and Sevastano's (2003) evidence of different countries' differing debt (in)tolerance. Actual debt ratios are below their maximum sustainable levels, as governments seeking further terms in office fear debt-induced default that may jeopardize their prospects for reelection. The difference between actual and maximum sustainable debt ratios creates a 'margin of safety' that allows governments to increase debt if necessary with little corresponding increase in default risk. The probability of default climbs precipitously once the margin of safety has been exhausted.

1 Introduction

Perhaps the defining characteristic of sovereign debt is its near total absence of enforcement mechanism: unlike the case for corporate debt, it is very difficult if not impossible for a creditor to seize the assets of a defaulting sovereign. Rogoff (1999, p. 31) consequently has deemed the question "why, exactly, are debtor countries willing to make repayments of any kind" to be "the crux of understanding international debt markets." Two answers to this question have been provided: the threat to deny a defaulting country further access to debt markets (Eaton and Gersovitz, 1981) and that to impose direct sanctions on the country (Bulow and Rogoff, 1989a, 1989b).[1] Building on these two answers, a number of papers have calibrated country debt-to-GDP ratios.[2] It is probably fair to say that they have fallen short of reproducing prevailing country debt levels.[3]

In the present paper, we revisit the issue of sovereign debt with a view to obtaining debt levels that are perhaps closer to the cross-sectional evidence than may so far have been obtained. For that purpose, we dispense with the two assumptions made by most existing papers, specifically a government with infinite horizon that defaults strategically, to replace these with the opposite assumptions of a government with horizon limited to its expected term in office that defaults only when unable to service its debt.[4]

We show that our alternative assumptions more naturally result in government debt levels in the vicinity of those observed in practice. A government whose horizon is limited to its expected term in office naturally neglects possibly negative consequences of government borrowing that occur beyond that term. A limited horizon government therefore can be expected to borrow more than its infinite horizon counterpart. High government demand for funds is met by high investor supply of funds, as investors who do not fear strategic default recognize that the limit to lending stems from the government's ability rather than willingness to service the debt: default occurs when government income falls short of debt service requirements. Investors further recognize that a government's ability to service existing debt depends on its ability to raise new debt: there is a bubble-like property to government debt. Investors base their lending decisions on government

[1] Possible sanctions have ranged from trade sanctions to outright occupation in the age of gunboat diplomacy.

[2] See the literature review in Section 2.

[3] Cohen and Villemot (2013) report debt-to-GDP ratios ranging from 1.5% to 30%, which their own work extends to 47%. Table 1 shows prevailing debt levels to be markedly higher for the vast majority of countries considered.

[4] These assumptions are justified below.

disposable income, that is, that part of government income that could if necessary be directed towards debt servicing: not all spending can be so directed, as the government must maintain some minimum level of services and investment. Government disposable income is a fraction of government income, itself a fraction of GDP. Thus, high ratios of debt to government income do not necessarily translate into high debt to GDP ratios.

We initially consider the case of a myopic government whose concern extends only to its current term in office. We characterize the government's maximum sustainable debt level and its associated default probability as functions of the mean and variance of growth in government disposable income, the ratio of government disposable income to government income and that of government income to GDP, and the risk-free rate. Maximum sustainable debt varies across countries; it can be viewed as a measure of a country's debt tolerance: countries that have lower maximum sustainable debt and/or higher associated default probability are, in the words of Reinhart, Rogoff, and Sevastano (2003), more debt intolerant. We then consider the case where the government's concern extends beyond the current term, perhaps because the government may be reelected to office. We show that the prospect of further terms in office induces the government to decrease borrowing below its maximum sustainable value, in order not to jeopardize through default the benefits of being in office during these further terms. The default probability correspondingly decreases.

We calibrate our model using IMF data over the period 1980-2011. Maximum sustainable debt levels can be surprisingly large, attaining 224% of GDP for Austria, 222% for France, and 216% for Sweden for example, under the admittedly arbitrary assumption that the ratio of government disposable income to government income is 40%.[5] These results may be due to relatively high mean growth rates (2.1% for Austria and 2.2% for Sweden, but 1.4% for France), relatively low growth volatilities (1.5% for Austria and 1.4% for France, but 2.2% for Sweden), and, last but not least, high ratios of government income to GDP (49% for Austria and for France, 56% for Sweden): perhaps not surprisingly governments that command a higher fraction of their countries' GDP can borrow more. These high maximum sustainable debt ratios are associated with relatively low maximum default probabilities (0.58% for Austria, 0.53% for France, and 0.91% for Sweden). These reflect the

[5]Note that even a halved disposable income to income ratio of 20% would still result in maximum debt ratios of 112% for Austria, 111% for France, and 108% for Sweden, well above was is generally considered sustainable. Importantly, sustainability is not the same as desirability; we return to that distinction in the Conclusion.

assumption of growth lognormality, analytically tractable as well as empirically warranted to some extent. Not all countries have high maximum sustainable debt levels and low default probabilities: Argentina's maximum sustainable debt level is 61% of GDP, reflecting the country's high growth volatility (6%) and low ratio of government income to GDP (28%); its mean growth rate over the period 1980-2011 was 2.5%. Argentina's maximum default probability is 3.01%, indicating that (relatively) low maximum sustainable debt levels may nonetheless be associated with (relatively) high maximum default probabilities.

Actual debt ratios are generally lower than maximum sustainable debt levels, reflecting the importance of the prospect of reelection. Thus, Austria, France, and Sweden had average debt ratios 64%, 50%, and 55% of GDP, respectively, over the period 1980-2011, ending with 72%, 86%, and 38% of GDP, respectively, in 2011. Associated default probabilities are essentially negligible. We interpret our findings as implying that a country whose actual debt ratio falls short of its maximum sustainable debt ratio enjoys a 'margin of safety' that affords the country the discretion to increase its debt to GDP ratio with little corresponding increase in its probability of default. This may explain why France, as well as the UK (2011 debt ratio 82%, maximum debt ratio 148%) and the US (2011 debt ratio 103%, maximum debt ratio 135%) have seen little if any increase in their costs of borrowing despite rather dramatic recent increases in their debt ratios. Things are very different once the margin of safety has been exhausted: Argentina had average debt ratio of 73% over the period 1980-2011, a mere 12% over its maximum sustainable level, yet the associated default probability was 82%. The same is true of Japan for example, whose maximum sustainable debt level is 106% of GDP with associated default probability 1.06%, but whose average debt ratio was 117% of GDP, with associated default probability 96%. The marked asymmetry between the probability of default's very slow increase below the maximum sustainable debt level and very fast increase above is a natural consequence of the trade-off involved in computing the maximum sustainable debt level under the assumption of lognormality: lenders equate the (infra)marginal benefit of increased repayment absent default with the marginal cost of an increased probability of default; where default is lognormally distributed, cost equals benefit around the point at which the probability of default starts its dramatic increase; debt levels that exceed the maximum sustainable level are therefore on or beyond that part of distribution where the probability of default increases very quickly.[6] The contrasting experiences of Argentina, which has defaulted on its debt, and Japan, which has not despite increasing its debt to 229% of GDP in 2011 (associated default probability 100%), suggest that our model captures only part of the debt phenomena at work.

As already noted, we replace the two assumptions of infinite government horizon and strategic default by the opposite assumptions of limited government horizon and default that occurs when government income falls short of debt service requirements—what Grossman and Van Huyck (1988) call 'excusable default.' We justify our decision on two grounds. First, we believe our assumptions are, on some dimensions at least, more realistic than the alternative assumptions. Regarding the length of the government's horizon, a theory of government debt predicated on the government's concern with developments that occur beyond the government's term of office clearly is at odds with the self-interest that Public Choice Theory for example attributes to government motives and behavior. Less rigorously but perhaps no less tellingly, conventional wisdom often holds that a government's horizon rarely extends beyond the next election, with the well-worn maxim *Après moi le déluge* considered accurately to reflect the attitude of the vast majority of governments in power. Regarding default, a theory of default based on strategic considerations is not consistent with Levy Yeyati and Panizza's (2011) evidence of governments' reluctance to default: governments default when they have no realistic option of servicing their debt, not when they deem the option to default to have a higher payoff than the option to service the debt. Levy Yeyati and Panizza (2011) attribute governments' reluctance to default to (i) governments' desire to be seen as engaging only in excusable default and (ii) governments' fear of losing office upon default. Tomz (2007) presents strong evidence that it is governments that engage in inexcusable default—default despite having income sufficient for debt service—that suffer the costs of default. Borensztein and Panizza (2008) and Malone (2011) find that governments that default see a marked decline in their prospects for reelection.[7]

Second, our assumptions make for simpler modeling than the alternative assumptions. Parsimony therefore should favor our assumptions over the alternative, in case the former should make possible the derivation of results no less satisfactory than those derived under the latter, at least

[6] To understand the intuition, it is helpful to represent the lognormal distribution by the succession of a very moderately sloped ray ending some distance before the mean, a very steeply sloped line segment ending some distance after the mean, and another very moderately sloped ray. The maximum sustainable debt level should be around the intersection of the first ray and the line segment, with higher debt levels being on the line segment or the second ray.

[7] For contrary evidence, see Foley-Fischer (2012).

in so far as concerns our primary variable of interest, specifically the level of government debt. Besides simplicity, our assumptions may in fact facilitate obtaining the desired result. Surely, lenders should be willing to lend more when default is excusable rather than strategic. Equilibrium levels of debt under excusable default therefore should be higher than under strategic default, thereby serving to attain the high levels of debt observed in practice. Our calibration results show this to be indeed the case.

The paper proceeds as follows. Section 2 briefly reviews the related literature. Section 3 presents the model. Section 4 considers the case of a myopic government to obtain a country's maximum sustainable debt ratio. Section 5 extends the analysis to the case of a government whose concern extends beyond the current term in office. Section 6 presents the data. Section 7 discusses the results of the calibration. Finally, Section 8 concludes.

2 Literature review

It is probably fair to say much of the recent literature on sovereign debt can be viewed as constituting a very rich tapestry weaved on the loom of Eaton and Gersovitz's (EG, 1981) seminal work.[8] Later work has quantified, refined, and extended EG's predictions, and endogenized some of what had been exogenous in EG. Thus Aguiar and Gopinath (2006) and Arellano (2008) have embedded the basic EG framework into the setting of a small open economy to study the interactions of default risk with output, consumption, the trade balance, interest rates, and foreign debt. Arellano (2008) ascribes the countercyclicality of interest rates and the current account to incomplete financial contracts. As interest and principal payments cannot be made to depend on output, the incentive to default is higher in recessions than in expansions. Interest rates consequently are lower in expansions, thereby inducing countries to borrow more when output is high. Borrowing finances imports, which deteriorate the current account. Aguiar and Gopinath (2006) incorporate a trend into the output process. They distinguish between the two cases of stable and volatile trends and show that only in the latter case can observed default frequencies be replicated in calibration. Where the trend in output is stable, there is little value to the insurance provided by access to international debt markets. A borrower in recession therefore has a strong incentive to default.

[8] The present literature review is by necessity rather limited in scope. For extensive literature reviews, see Panizza, Sturzenegger, and Zettelmeyer (2009), Reinhart and Rogoff (2009), and Sturzenegger and Zettelmeyer (2006).

The interest rate schedule consequently is extremely steep and borrowing does not extend to the range where default occurs. Where in contrast the trend in output is volatile, insurance is valuable and the incentive to default is weakened. The interest rate schedule is less steep and borrowing extends to the range where default occurs.

Mendoza and Yue (2012) have endogenized output and the collapse in output that accompanies default. Theirs is a general equilibrium model in which domestic firms borrow internationally to finance their purchase of foreign inputs. A sovereign default jeopardizes firms' access to foreign working capital loans, thereby forcing the firms to substitute domestic inputs for the previously purchased foreign inputs. As the former are imperfect substitutes for the latter, TFP declines and the effects of the negative output shock that triggered default are amplified. Cuadra and Sapriza (2008) have considered the role of political risk. They show that political instability (one party may lose power to another party) and political polarization (different parties represent different constituents with differing interests) combine to increase borrowing by decreasing the importance a party in power attaches to the future. The negative consequences of borrowing are lessened when shared with another party that has other constituents. A positive consequence of borrowing is to 'tie the other party's hands,' thereby preventing that party from lavishing its constituents with debt-financed favors should the party come to power.[9]

Yue (2009) and Benjamin and Wright (2009) have considered the role of renegotiation in default. Yue (2009) considers Nash bargaining under symmetric information. Disagreement payoffs are zero for creditors and the autarkic payoff for the defaulting country. Yue (2009) shows that the parties bargain to a reduced level of debt that does not depend on the defaulting country's original debt: the parties 'let bygones be bygones.' Haircuts therefore are increasing in the defaulting country's debt. They are decreasing in the country's output: countercyclical interest rates increase the payoff for the country to rejoining international debt markets; they increase the bargaining surplus that is shared by the parties. Benjamin and Wright (2009) note that the period to the resolution of default extends over many years. They attribute the delay in default resolution to the requirement that the defaulting country's commitment to servicing post-resolution debt be credible. As the incentive to default generally decreases in output, credibility requires that the defaulting country's

[9] Our analysis in Section 5 shares much with that of Cuadra and Sapriza (2008), but not the assumptions of strategic default or infinite government horizon.

output recovers from the low level that likely prompted default in the first place. This is often a protracted process. That the country emerges from default only after output has recovered provides an explanation for the otherwise puzzling observation that default resolution often results in post-resolution debt that is no lower than the original, pre-default debt.

Hatchondo and Martinez (2009) and Chatterjee and Eyigungor (2012) have considered the role of debt maturity: when not all debt is retired every period, the issuance of new debt serves to dilute the value of existing debt; lack of commitment creates a 'prisoner's dilemma' that results in increased government borrowing at higher interest rates. While short-term debt therefore should dominate long-term debt, this need not be true where self-fulfilling rollover crises may occur (Chatterjee and Eyigungor, 2012). Fink and Scholl (2011) have considered the role of conditionality. They show that international financial institution (IFI) involvement may increase rather than decrease interest rates, by inducing additional borrowing on the part of a government that expects to benefit from IFI support.

Cohen and Villemot (2013) have noted the difficulty of existing models simultaneously to match the first moments of debt and default probabilities: high default costs that make possible the matching of debt ratios preclude that of default probabilities; low default costs have the opposite effect.[10] Building on Levy Yeyati and Panizza's (2011) finding that output contractions generally precede rather than follow default, Cohen and Villemot (2013) have developed a model in which the cost of default is borne 'in advance.' Governments in such case do not have the incentive to stave off a default whose cost they have already borne. As already noted, our work departs from the EG assumptions: there is neither strategic default nor infinite government horizon in our model.[11]

We conclude the present section by noting that, unlike the assumption of strategic default, the assumption of excusable default is not subject to the well-known Bulow-Rogoff critique (Bulow and Rogoff, 1989a, 1989b), whereby exclusion from debt markets alone fails to deter default because a defaulting government can use the amount otherwise to be reimbursed to purchase an insurance contract that provides the same risk sharing as does government borrowing. A government that

[10] See in particular Table 1 in Cohen and Villemot (2013).

[11] Some previous work has maintained one but not the other EG assumption. For example, Catão and Kapur (2004) maintain the assumption of strategic default but dispense with that of infinite horizon. They focus on the effect of macroeconomic volatility on government debt. Conversely, Bi and Leeper (2012) maintain the assumption of infinite horizon but dispense with that of strategic default. They characterize the fiscal limit that arises from the dynamic Laffer curve.

has excusably defaulted has no income with which to purchase the insurance contract. It further stands to lose office.

3 The model

Assume for simplicity a government's term in office lasts a single year and consider a government that is in office in year t. Let y_t denote the government's disposable income in year t; b_t denote the proceeds from issuing debt in year t, expressed as a fraction of government disposable income y_t; d_t denote the face value of that debt, again expressed as a fraction of government disposable income y_t but payable in year $t+1$; g denote the gross rate of growth in government disposable income, from y_t to y_{t+1}, distributed $\ln(g) \sim N(\mu, \sigma^2) \equiv F(g)$, $f(g) \equiv F'(g)$; and r denote the risk-free interest rate.[12] The maximum amount the government can borrow is

$$b_t y_t = \frac{\Pr\left[(1+b_{t+1})\, y_{t+1} > d_t y_t\right] d_t y_t + \int_0^{d_t y_t/(1+b_{t+1})} y_{t+1} dF(y_{t+1})}{1+r} \tag{1}$$

Default occurs in year $t+1$ when the sum of government disposable income in year $t+1$ (y_{t+1}) and the amount the new government can borrow in that year ($b_{t+1}y_{t+1}$) is not sufficient to service the debt raised in year t ($d_t y_t$). We assume lenders can appropriate the entirety of government disposable income in default, but that no new borrowing is possible in such case: lenders do not 'throw good money after bad;' there is a 'sudden stop.'[13] This is unlike the case of no-default, in which proceeds from new borrowing can be used to service existing debt. There is thus a 'bubble-like' property to government debt: the debt the government can raise in year t, $b_t y_t$, depends on the debt the government can raise in year $t+1$, $b_{t+1}y_{t+1}$.

[12] We distinguish between government (total) income and government disposable income and express debt as a fraction of the latter because even the most creditor-harried governments are unlikely to allocate all income to debt servicing. While a government can and generally will cut non-debt servicing spending to avoid defaulting on its debt, it is difficult to envision a government reducing all spending on defense, education, health, and law and order to zero.

[13] We assume without loss of generality that default lasts only a single period.

Rearranging (1), we have

$$
\begin{aligned}
b_t &= \frac{1}{1+r}\left[\Pr\left[\frac{y_{t+1}}{y_t} > \frac{d_t}{1+b_{t+1}}\right] d_t + \int_0^{d_t y_t/(1+b_{t+1})} \frac{y_{t+1}}{y_t} dF\left(y_{t+1}\right)\right] \\
&= \frac{1}{1+r}\left[\Pr\left[g > \frac{d_t}{1+b_{t+1}}\right] d_t + \int_0^{d_t/(1+b_{t+1})} g dF\left(g\right)\right] \\
&= \frac{1}{1+r}\left[\left[1 - F\left(\frac{d_t}{1+b_{t+1}}\right)\right] d_t + \int_0^{d_t/(1+b_{t+1})} g dF\left(g\right)\right] \qquad (2)
\end{aligned}
$$

Using the lognormality of $F(.)$ and defining

$$
z_t \equiv \frac{\ln\left(\frac{d_t}{1+b_{t+1}}\right) - \mu}{\sigma} \qquad (3)
$$

we can rewrite (2) as[14]

$$
b_t = \frac{e^{\mu}}{1+r}\left[e^{\sigma z_t}\left(1+b_{t+1}\right)\left[1-\Phi\left(z_t\right)\right] + e^{\frac{\sigma^2}{2}}\Phi\left(z_t - \sigma\right)\right] \qquad (4)
$$

where $\Phi(.)$ denotes the standard normal cdf.

We have thus far assumed that the funds at the government's disposal in year t, $y_t + b_t y_t$, suffice to service the debt $d_{t-1}y_{t-1}$ that the government has 'inherited' from the government that was in office in year $t-1$ (recall that a government remains in office only a single term, that is, only a single year): $y_t + b_t y_t \geqslant d_{t-1}y_{t-1}$. If that should not be the case, and if $y_t + b_t y_t < d_{t-1}y_{t-1}$, then the government defaults during its term of office in year t. We assume that the government loses office upon default. This assumption plays no role in Section 4, but will play an important role in Section 5.

[14]To transform (2) into (4), rewrite $d_t/\left(1+b_{t+1}\right)$ as $\exp\left(\mu+\sigma z_t\right)$, $F\left(d_t/\left(1+b_{t+1}\right)\right)$ as $\Phi\left(\left(\ln\left(d_t/\left(1+b_{t+1}\right)\right)-\mu\right)/\sigma\right) = \Phi\left(z_t\right)$, and $\int_0^{x_t} g dF\left(g\right)$ as

$$
\int_0^{d_t/\left(1+b_{t+1}\right)} g d\Phi\left(\frac{\ln\left(g\right)-\mu}{\sigma}\right) = \int_{-\infty}^{z_t} \exp\left(\mu+\sigma z_g\right) d\Phi\left(z_g\right) = e^{\mu+\frac{\sigma^2}{2}}\Phi\left(z_t-\sigma\right)
$$

where $z_g \equiv \left(\ln\left(g\right)-\mu\right)/\sigma$.

4 Myopic government and maximum sustainable debt

Consider a myopic government whose horizon does not extend beyond its current term in office, perhaps because it is all but certain to lose power at the next election. The government naturally raises all the debt it can: the government need not concern itself with default—except insofar as default affects the amount the government can borrow—because it will not be in office to be confronted with the consequences of default.[15] We define maximum sustainable borrowing to be the maximum amount the government can borrow on a sustained basis, and maximum sustainable debt the corresponding amount owed by the government.

A myopic government maximizes present borrowing b_t given lenders' expectation of future borrowing b_{t+1}[16]

$$b_t = \max_z \frac{e^{\mu}}{1+r}\left[e^{\sigma z_t}\left(1+b_{t+1}\right)\left[1-\Phi\left(z_t\right)\right]+e^{\frac{\sigma^2}{2}}\Phi\left(z_t-\sigma\right)\right] \equiv \tau\left(b_{t+1}\right) \tag{5}$$

A Rational Expectations Equilibrium (REE) is a sequence of b_t's that satisfies $b_t = \tau\left(b_{t+1}\right)$. All REE are unbounded, except maximum sustainable borrowing $b_t = b_{t+1} = b_M$ with $b_M = \tau\left(b_M\right)$.[17] We therefore seek the fixed point

$$b_M = \max_z B\left(b_M, z\right) \tag{6}$$

where

$$B\left(b_M, z\right) \equiv \frac{e^{\mu}}{1+r}\left[e^{\sigma z}\left(1+b_M\right)\left[1-\Phi\left(z\right)\right]+e^{\frac{\sigma^2}{2}}\Phi\left(z-\sigma\right)\right] \tag{7}$$

We show[18]

Proposition 1 *If $E\left[g\right] < 1+r$, $\tau\left(b\right) = \max_z B\left(b,z\right)$ is a contraction mapping and (6) has a unique fixed point, b_M.*

The result recalls the condition for the convergence of a growing dividend stream: only a sufficiently high interest rate precludes the mortgaging of all future disposable income. Figure 1 shows the determination of b_M graphically. The condition $E\left[g\right] < 1+r$ ensures that the slope of

[15] Recall from Section 3 that debt has maturity one period.

[16] For simplicity, we refer to b_t as 'borrowing' rather than 'borrowing as a fraction of government disposable income,' more exact but also longer. We likewise refer to d_t as debt.

[17] We use the subscript M for maximum.

[18] All proofs are in the Appendix.

$\tau(b)$ remains below unity; there is no intersection between the curve and the line beyond b_M.

We denote $z_M \equiv \arg\max_z B(b_M, z)$ and show

Proposition 2 *Maximum sustainable government borrowing b_M is increasing in the mean growth rate μ, decreasing in growth rate volatility σ for $z_M < 0$, and decreasing in the risk-free interest rate r. The corresponding probability of default $\Phi(z_M)$ is decreasing in the mean growth rate μ, increasing in growth rate volatility σ for $z_M < 0$, and decreasing in the risk-free interest rate r.*

The results are intuitive. A government that is expected to see its disposable income grow faster can borrow more, for it is expected to have more income with which to service its debt. In contrast, a government whose disposable income growth is more volatile can borrow less, for the greater likelihood of low income realizations increases the probability of default, thereby decreasing lenders' willingness to lend to the government.[19] A government can borrow less when the risk-free interest rate is high, for a high risk-free rate raises lenders' opportunity cost of lending to the risky government, thereby decreasing lenders' willingness to lend to the government. The higher risk-free rate increases the interest rate the government must pay on its debt (see Proposition 4 below), thereby increasing the probability of default.

What is true of the probability of default at maximum sustainable borrowing extends to the probability of default at any level of borrowing. Consider a government that will owe dy_t in period $t+1$ as a result of having borrowed by_t in period t. Denote $\Phi(z)$ the corresponding probability of default. We have

Proposition 3 *The probability of default $\Phi(z)$ is decreasing in the mean growth rate μ, increasing in growth rate volatility σ for $z < 0$, and increasing in the risk-free interest rate r.*

By analogy to maximum sustainable borrowing b_M, we define maximum sustainable debt d_M. Denoting $z_M = \arg\max_z B(b_M, z)$, we have from $d_t/(1 + b_{t+1}) = \exp(\mu + \sigma z_t)$ in footnote 14

$$d_M = \exp(\mu + \sigma z_M)(1 + b_M) \tag{8}$$

[19] More volatile disposable income growth also results in a greater likelihood of *high* income realizations. The condition $z_M < 0$ ensures that the detrimental effect of volatility on debt dominates. It amounts to the reasonable condition that the probability of default be less than 1/2.

We further define the interest rate $R_M \equiv d_M/b_M$ and the expected loss given default $\overline{LGD}_M$. These are[20]

$$R_M \equiv \frac{d_M}{b_M} = \frac{\exp(\mu + \sigma z_M)(1 + b_M)}{b_M} = \frac{\exp(\mu + \sigma z_M)\phi(z_M)}{\sigma[1 - \Phi(z_M)]} \tag{9}$$

and[21]

$$\begin{aligned} \overline{LGD}_M &\equiv \frac{d_M y_t - \int_0^{d_M y_t/(1+b_M)} y_{t+1} \frac{dF(y_{t+1})}{F(d_M y_t/(1+b_M))}}{d_M y_t} \\ &= 1 - \frac{E[\exp(\sigma z_g) | z_g < z_M]}{\exp(\sigma z_M)(1 + b_M)} \end{aligned} \tag{10}$$

We show

Proposition 4 *Maximum sustainable debt d_M is increasing in the mean growth rate μ and decreasing in the risk-free interest rate r for $z_M < 0$. The interest rate R_M is increasing in the risk-free interest rate r for $z_M < 0$.*

Not all comparative statics can be determined because the direct effects of μ, σ, and r on d_M, R_M, and $\overline{LGD}_M$ often are often counteracted by their indirect effects through z_M or b_M. Consider for example maximum sustainable debt d_M, which intuition and the result $\partial b_M/\partial\sigma < 0$ in Proposition 2 suggest should be decreasing in growth rate volatility σ. That such is not the case is due to the offsetting role of the probability of default $\Phi(z_M)$, the increase in which requires an increase in debt to be repaid absent default d_M as compensation for the larger probability of default in which partial payment only is received. Similar considerations apply to the interest rate R_M and the expected loss given default $\overline{LGD}_M$.

An alternative characterization of d_M will prove useful in interpretating our calibration results below. Use (2) and (9) to define

$$B(d) \equiv \frac{1}{1+r}\left[\left[1 - F\left(\frac{d}{1+b_M}\right)\right] d + \int_0^{d/(1+b_M)} g dF(g)\right] \tag{11}$$

and

$$R(d) \equiv \frac{d}{B(d)} \tag{12}$$

[20] The last equality in (9) uses $(1 + b_M)/b_M = \phi(z_M)/\{\sigma[1 - \Phi(z_M)]\}$ from (21) in the Proof of Proposition 2.
[21] The equality in (10) is derived in the Appendix.

respectively. By analogy to $z_M = \arg\max_z B(b_M, z)$, we have $d_M = \arg\max_d B(d)$. Using $B'(d_M) = 0$, we can write

$$R'(d_M) = \frac{R(d_M)}{d_M}$$

In words, maximum sustainable debt equates the marginal and average interest rates. The average interest rate therefore reaches its minimum at maximum sustainable debt d_M.

5 Beyond the current term

We now consider the case where a government serving its first term in office has, in the absence of default, an exogenous probability θ of winning the next election. Should it win that election, the government will serve another term in office, at the end of which it will again have an exogenous probability θ of winning the next election, again in the absence of default. The government loses office upon default. Our purpose in the present section is to examine how the prospect of reelection and the likelihood of further terms in office alters the government's borrowing as compared to the case of a myopic government that serves a single term only.

Consider a government that has inherited debt $d_{t-1}y_{t-1}$ and has disposable income y_t in year t. The government has value function $V(.)$ such that

$$V(d_{t-1}y_{t-1}, y_t) = \max_{b_t} u(y_t + b_t y_t - d_{t-1}y_{t-1}) + \frac{\theta}{1+r} E[V(d_t y_t, y_{t+1})] \tag{13}$$

where $u(.)$ denotes the government's utility function.[22] The government recognizes that its choice of borrowing in year t, $b_t y_t$, determines the debt it owes in year $t+1$, $d_t y_t$. The expectation is over g as $y_{t+1} = y_t g$.

In order to make our problem tractable, we assume that the government has CRRA utility with RRA coefficient a

$$u(c) = \frac{c^{1-a}}{1-a} \tag{14}$$

[22] Note that there was no need to introduce the government's utility function in Section 5, because the single term of office implied that the government would maximize borrowing regardless of its specific utility function.

This makes the value function similarly CRRA with RRA coefficient a

$$V\left(d_{t-1}y_{t-1}, y_t\right) = V\left(d_{t-1}\frac{y_{t-1}}{y_t}, 1\right) y_t^{1-a} \equiv v\left(d'_{t-1}\right) y_t^{1-a} \tag{15}$$

where $d'_{t-1} \equiv d_{t-1}\left(y_{t-1}/y_t\right)$ represents the debt the government owes in year t, expressed as a fraction not of disposable income in year $t-1$ in which the debt was raised, but as a fraction of disposable income in year t in which the debt is to be repaid. Put differently, $d'_{t-1}y_t = d_{t-1}y_{t-1}$ is the debt owed in year t as repayment for the amount $b_{t-1}y_{t-1}$ borrowed in year $t-1$.

Problem (13) can be rewritten[23]

$$v\left(d'_{t-1}\right) = \max_{b_t} u\left(1 + b_t - d'_{t-1}\right) + \frac{\theta}{1+r} E\left[g^{1-a} v\left(\frac{d_t}{g}\right)\right] \tag{16}$$

By analogy to (1) in Section 4, note that the relation between b_t (fraction borrowed) and d_t (fraction owed) is given by

$$b_t y_t = \frac{\Pr\left[\left(1+b_M\right) y_{t+1} > d_t y_t\right] d_t y_t + \int_0^{d_t y_t/(1+b_M)} y_{t+1} dF\left(y_{t+1}\right)}{1+r} \tag{17}$$

Equation (17) differs from (1) in replacing b_{t+1} by b_M: the assumption of excusable default implies that the government will borrow as much as it can in order to stave off default.[24] The stationary REE satisfies

$$v\left(d'\right) = \max_{x} u\left(1 + B\left(d\right) - d'\right) + \frac{\theta}{1+r} E\left[g^{1-a} v\left(\frac{d}{g}\right)\right] \tag{18}$$

with $B\left(d\right)$ defined in (11). Note the similarity between (2) in Section 4 and (**??**). Indeed, the solutions $b^* \equiv B\left(d^*\right)$ and d^* reduce to b_M, and d_M for $\theta = 0$.

Further insights into (18) may be gained by examining the relation between d' (debt to be repaid in the current period as a fraction of current period government disposable income, 'old' debt) and d (debt to be repaid in the next period as a fraction of current period government disposable income, 'new' debt), shown in Figures 1 and 2, for the two cases of risk-neutral government ($a = 0$) and risk-averse government ($a > 0$), respectively. Under risk-neutrality (Figure 2), the government

[23] Divide (13) by y_t^{1-a} and use (14) and (15). The condition for the existence and uniqueness of the value function $v\left(.\right)$ can be shown to be $\theta E\left[g^{1-a}\right] < 1 + r$.

[24] Corollary 1 below shows that maximum sustainable borrowing remains at b_M.

maximizes the present value of a stream of disposable income that ends either with default or with the failure to be reelected. A government that expects to be reelected with certainty absent default ($\theta = 1$) finds it beneficial to decrease the probability of default to zero, thereby receiving the entire stream with certainty. Where debt owed in the current period is less than current disposable income ($d' < 1$), the government repays that old debt in its entirety and raises no new debt ($d = 0$), thereby achieving the desired decrease of the default probability to zero. As debt owed increases beyond current disposable income ($d' > 1$), the government is forced to borrow anew ($d > 0$) in order to service the debt it owes. The probability of default is as low as can be given debt owed. Maximum debt ($d = d_M$) is attained where debt owed equals the sum of current disposable income and maximum amount that can be borrowed ($d' = 1 + b_M$). The government defaults beyond. Where there is some non-zero probability that the government loses the next election ($\theta < 1$), the government is now willing to borrow in the present period ($d = d^*$): it enjoys the benefits of such borrowing with probability one, yet will be confronted with its costs with probability less than one, for it will be voted out of office with probability greater than zero. Again, there are circumstances ($d' > d'^*$) where the government borrows more than it otherwise deems desirable ($d > d^*$), just for the purpose of staving off default.

Under risk-aversion (Figure 3), the government wishes to equalize the marginal utility of income across periods, net of debt proceeds and repayments. The expectation of growing disposable income ($\mu > 0$) implies that there is always some borrowing ($d > 0$), in order to bring forward in time some of that disposable income growth. As in the case of risk-neutrality, borrowing is lowest where the probability of reelection is highest at unity ($\theta = 1$), for there no opportunity for the government to 'escape' being confronted with default through electoral defeat in such case. New debt (d) increases in old debt (d'), as the government seeks to make up through new debt for old debt's higher claim on current disposable income. Again, there is default where old debt is larger than the sum of current disposable income and maximum amount that can be borrowed ($d' = 1 + b_M$).

We show

Proposition 5 *Government borrowing $b^* \equiv B(d^*)$, debt owed d^*, and the probability of default $F(d^* / (1 + b_M))$ decrease in the probability of reelection θ.*

Corollary 1 *$b^* \leqslant b_M$, $d^* \leqslant d_M$, and $F(d^*/(1+b_M)) \leqslant F(d_M/(1+b_M))$ for $\theta \geqslant 0$ with equality at $\theta = 0$.*

The intuition is simple: the government values further terms in office, which it would forego in case it were to default; the government therefore decreases the probability of default by decreasing the amount it borrows. The higher the probability of reelection, the higher the expected value of further terms in office, and the greater therefore the incentive to decrease the probability of default by decreasing borrowing.

6 Data

We use country GDP data from IMF statistics to compute mean growth in GDP μ and volatility σ over the period 1980-2011. There are 186 countries and 32 years, but countries that came into existence during that period naturally have data for fewer years. South Sudan for example has GDP data for the single year 2011 in which it became independent. Other countries have missing data because of wars (Afghanistan, Iraq) or other, much less tragic but also less obvious reasons (Malta for example has GDP data for 12 years). The same issues arise for the values of the debt-to-GDP and government income-to-GDP ratios, also obtained from IMF statistics. There are no estimates of the probability of reelection θ, nor of the ratio of government disposable income to government income. We keep our analysis of the former on a qualitative level and, in a rough and ready way, conjecture the former to be around 0.4. In other words, we conjecture that a maximum of 60% of government income is allocated to essential spending that cannot be cut, at least not on a sustainable basis; government debt is therefore serviced out of the remaining 40% of government income.[25] We use the mean US 30-year bond yield over the period for the risk-free interest rate r. In view of Proposition 1, we exclude those countries for which $E[g] > 1 + r$.

The data for selected countries is shown in Table 1. There is a wide range of debt ratios, as well as mean growth rates and volatilities. The country with the lowest average debt ratio over the period 1980-2011 was Chile at 11%, followed by Australia at 20%. Chile retains its first place in 2011, whilst Australia cedes its second place to Russia, which will be recalled to have defaulted on its debt in 1998 and 1999. Other countries in our list that defaulted on or restructured their debt over

[25] As noted in the Introduction, the overall gist of our results remains where the 0.4 ratio is halved.

the period 1980-2011 are Argentina, Brazil, Ecuador, Egypt, Greece, Iceland, Indonesia, Mexico, Peru, Romania, Turkey, Ukraine, Uruguay, and Venezuela, consistent with Reinhart and Rogoff's (2009) observation that sovereign defaults are a common occurrence. Default often results in a 2011 debt ratio that is lower than its 1980-2011 average. This is most clearly the case for Argentina, whose debt ratio declined from a 1980-2011 average of 73% to 45% in 2011, and Indonesia, where the decline was from 49% to 24%. It is not only defaulting countries that saw their debt ratios decline: Belgium saw its debt to GDP ratio decline from a 1980-2001 average of 109% to 97%, Sweden from 55% to 38%; Denmark, Israel, New Zealand, and Switzerland saw somewhat smaller declines. Most industrialized countries saw their debt ratios increase, though, most notably Greece (from 85% to 165%), Japan (from 118% to 229%), and, on a smaller yet still sizeable scale, Iceland (from 45% to 99%), the United Kingdom (from 46% to 82%), and the United States (from 65% to 103%).

In line with Catão and Kapur's (2004) argument and findings, countries with more volatile growth rates tend to have lower debt ratios. The countries with volatility equal to or greater than 6% are Argentina (6.00%), Peru (6.10%), Russia (6.84%), Ukraine (10.59%), and Venezuela (6.77%). Argentina's 1980-2011 average excluded, these countries have had low debt ratios, ranging from Russia's 12% to Venezuela's 47%, both in 2011. There are many countries with equally low debt ratios, though, despite having markedly less volatile growth rates. South Africa for example has volatility 2.41% yet debt ratio 39% in 2011, only slightly above Ukraine's 36% in that same year. This may reflect the ambiguous effect of volatility on debt, as well as the influence of other factors.[26]

Not least among these other factors is the ratio of government income to GDP. By and large, countries for which that ratio is higher tend to have higher debt ratios. Returning to our comparison of South Africa and Ukraine, note that the former has government income to GDP ratio of 27%, the latter of 39%: Ukraine's government can command more of its country's GDP than can its South African counterpart; it correspondingly can borrow more. Another factor is the mean growth rate. The result $\partial d_M/\partial\mu > 0$ from Proposition 4 notwithstanding, slower growing countries tend to have

[26] That volatility has an ambiguous effect on debt is suggested by our failure to sign $\partial d_M/\partial\sigma$ in Section 4. This (non-)result is only suggestive as it pertains to maximum sustainable debt d_M rather than actual debt considered in the present section.

higher debt ratios.[27] Still, the slowest growing country, the Ukraine at -0.8%, has a relatively low debt ratio of 36% in 2011; the fastest growing country, Vietnam at 6.3%, has a moderate debt ratio of 50% in 2011. Yet another factor affecting debt ratios is the probability of reelection. Whilst little quantitative can be said, note that democracies, whose governments presumably have shorter expected terms in office, tend to have higher debt ratios; this is rather consistent with Proposition 5.

7 Calibration results

Table 1 shows the calibration of maximum sustainable debt d_M. Note that two countries, Greece and Japan, have actual debt levels that are well in excess of their respective d_M. Such discrepancy may at least partially account for Greece's 2011 default; it suggests that further debt writedowns may be in order. That there have been no similar developments in Japan is, to us at least, a matter of some surprise.[28] An obvious question is how a country can borrow well above its maximum sustainable debt level. One does not have to agree with our exact calculation of d_M to believe that a country must have some maximum sustainable debt ratio; any mistake in our calculation of d_M would have to be very large indeed for that sustainable ratio to be in the range of Greece and Japan's actual debt ratios. One possible answer is deceptive statistics: Greece is said to have falsified its debt numbers for the better part of a decade.[29] This answer does not apply to Japan. Another is a large currency devaluation that inflates foreign debt in local currency terms, or a large negative shock to GDP. Again, this does not apply to Japan. It does, however, apply to many of the countries that defaulted over the 1980-2011 period we consider.

Sturzenegger and Zettelmeyer (2006) provide detailed case studies of the defaults of Argentina, Ecuador, Russia, Ukraine, and Uruguay, The effect of is local currency devaluation is most clearly seen in Uruguay. In June 2002, in no small part in response to Argentina's 2001 default and the Argentine peso's ensuing devaluation, Uruguay floated the Uruguayan peso. A 50% devaluation followed, which, by Sturzenegger and Zettelmeyer's (2006) calculation, accounted for more than

[27] As for our failure to sign $\partial d_M/\partial \sigma$, the result $\partial d_M/\partial \mu > 0$ is only suggestive, for it too pertains to maximum sustainable debt d_M rather than actual debt.

[28] We realize Japan is on many counts a special case, not least because its large private savings are invested overwhelmingly if indirectly in Japan Government Bonds. It is nonetheless difficult to view Japan's current debt levels and policies as sustainable (Hoshi, 2011; Hoshi and Ito, 2012).

[29] See for example Story, Thomas, and Schwartz (2010).

half the increase in Uruguay's debt-to-GDP ratio from 54% at end 2001 to 94% at end 2002.[30] With Uruguay's d_M at 80%, what had been a very comfortable debt ratio (near zero probability of default at 54% debt ratio) became a very heavy burden (rather coincidentally, 94% probability of default at 94% debt ratio). Uruguay defaulted in May 2003.

Uruguay's default occurred after it had devalued. In contrast, Argentina's default occurred before it abandoned its peg to the US dollar, suggesting that not only the reality but also the expectation of a devaluation that would increase the debt ratio beyond its sustainable level may suffice to trigger default. At least since the devaluation of the Brazilian real in 1999, Argentina's peso had been under pressure, with several analysts questioning the viability of its peg to the US dollar. When Argentina defaulted on December 24, 2001, its debt-to-GDP ratio was 53%, well below its d_M at 61%. Yet, within slightly more than a month, the peso was devalued; it was to go from parity with the dollar to a ratio of 3.7:1 in the space of six months, before settling down at around 3:1. This decline was reflected in Argentina's debt-to-GDP ratio, which was to increase to 150% at end 2002; Sturzenegger and Zettelmeyer (2006) attribute the bulk of this increase, 60%, to the peso's devaluation. At 113% debt-to-GDP ratio, Argentina's default probability was 100%; investors were not mistaken in fearing that devaluation would result in near-certain default.[31]

Much the same phenomenon appears to be at work in the cases of Russia and Ecuador. In Russia, whose d_M is 72%, successive devaluations helped transform a debt-to-GDP ratio of 54% (near zero default probability) at end 1997 into 68% (0.92% default probability) at end 1998 and 90% (92% default probability) at end 1999. The sharp fall in oil prices in the wake of the 1997 Asian crisis decreased government revenues and may have prompted Russia's default on its domestic debt in August 1998. Russia concurrently conducted a first devaluation, presumably because falling oil prices limited the foreign currency available to defend the ruble's peg. As noted by Sturzenegger and Zettelmeyer (2006, p.104), "the devaluation led to a sharp rise in the share of debt to GDP and made it increasingly difficult to remain current on external debt payments." Russia was placed in default on its external debt in January 1999. Between 1997 and 1999, the ruble had gone from an

[30] There was a concurrent 11% decline in GDP.

[31] We obtain 113% by adding to the initial debt-to-GDP ratio (53%) the increase that is attributable to devaluation (60%) . We realize that causality runs in both directions, from default to devaluation as well as from devaluation to default. Yet, it would be difficult to argue that devaluation would have been avoided had default not occured: Argentina had a large trade deficit which it was having difficulty financing; the peso's peg to the dollar was increasingly being questioned.

average of 5.8 to the dollar to 24.6:1; it was to reach and remain in the high twenties for the next few years. In Ecuador, which defaulted on its debt in January 1999, real depreciation contributed to nearly two-thirds of the increase in the debt-to-GDP ratio from 67% (near zero default probability) at end 1998 to 101% (near certainty of default) at end 1999. Ecuador's d_M is 79%.

Only in Ukraine did devaluation not play an important part in the country's September 1998 default: Ukraine's debt-to-GDP ratio was 33% in 1997 (near zero default probability), 37% in 1998 (near zero default probability), and its maximum sustainable value d_M of 52.8% in 1999 (6.81% default probability). Thus, we are able to account neither for Japan's resilience to a debt-to-GDP ratio well above its maximum sustainable value nor for Ukraine's default at a debt-to-GDP ratio at most equal to its maximum sustainable value.[32]

Maximum sustainable debt ratios vary widely across countries, as do these levels' corresponding default probabilities. Austria has the highest maximum sustainable debt ratio (224%), Peru the lowest (44%); at their respective maximum sustainable debt ratios, Ukraine has the highest probability of default (6.81%), France the lowest (0.54%). Consistently with Reinhart, Rogoff, and Sevastano (2003), different countries thus have differing debt tolerance. Reinhart, Rogoff, and Sevastano (2003, p. 1) find that "debt-intolerant countries tend to have weak fiscal structures," Catão and Kapur (2004) that they have higher macroeconomic volatility. Consistently with these findings, the maximum sustainable debt ratio d_M is negatively correlated with growth rate volatility σ (correlation coefficient -0.75) and positively correlated with the ratio of government income to GDP (correlation coefficient 0.81); in the spirit of Besley and Persson (2011), that ratio may be viewed as a measure of the country's fiscal strength.[33] Interestingly, there is very little correlation between the maximum sustainable debt ratio d_M and the mean growth rate μ (correlation coefficient -0.05): the maximum sustainable debt ratio depends much more on the volatility of growth than on its mean.[34]

Table 1 also shows the calibration of the 'minimum' debt ratios d_m, those that correspond to the cases $\theta = 1$ and $a \longrightarrow 0$ in which, as argued in Section 5, a government wishes to minimize

[32] The 6.81% default probability at $d_M = 52.8\%$ admittedly is not negligible. Still, Ukraine's growth in 1998 (-1.9%) and 1999 (-0.2%) was not so low as to suggest that default was due to a low growth realization.

[33] Besley and Persson (2011) associate state capability with the ability to raise taxes, income tax in particular.

[34] Note that the negative correlation between maximum sustainable debt d_M and mean growth rate μ does not necessarily contradict the result $\partial d_M/\partial\mu > 0$ from Proposition 4: the simple correlation fails to keep other determinants of maximum sustainable debt constant.

borrowing in order not to jeopardize through default futures terms in office. That $d_m > 0$ despite $\theta = 1$ and $a \longrightarrow 0$ suggests either (i) that d_m is discontinuous at $a = 0$ (compare d at $d' = 0$ for $\theta = 1$ in Figures 1 and 2) or (ii) that existing debt is such that even a government that wishes completely to pay down debt is forced to engage in some borrowing in order service existing debt ($1 < d' < 1 + b_M$ in Figure 2). Every d_m has associated default probability effectively zero, as is probably to be expected at the low levels that characterize d_m. More surprisingly perhaps, country default probabilities increases very slowly to reach default probabilities that for the most part are quite reasonable at d_M; they then increase quasi exponentially. This can be seen in Table 1 for Argentina, Greece, and Japan; it was already apparent in our calibration of default probabilities at different debt levels in our discussion of the Argentine, Uruguayan, Russian, and Ecuadoran defaults.[35]

To understand the asymmetry between the probability of default's very slow increase below the maximum sustainable debt level and very fast increase above, it is helpful to return to the discussion at the end of Section 4. We know the average interest rate $R(d)/d$ reaches its minimum at maximum sustainable debt d_M. While the interest rate $R(d)$ naturally always increases in debt d, the decrease in the average interest rate for $d < d_M$ and increase for $d > d_M$ implies that the interest rate must increase slowly for $d < d_M$ and rapidly for $d > d_M$. As the interest rate and the default probability are positively related, the slow increase before d_M and rapid after is true of the probability of default too.[36] The asymmetry is compounded by the very low volatility of growth, which make the lognormal distribution rather flat for low d ($d < d_M$) and very steep for d around the mean ($d > d_M$).

In summary, a country whose actual debt ratio falls short of its maximum sustainable debt ratio enjoys a 'margin of safety' that affords the country the discretion to increase its debt to GDP ratio with little corresponding increase in its probability of default. The probability of default climbs precipitously once the margin of safety has been exhausted.

[35] Again, Ukraine is a case apart.

[36] That the interest rate and the default probability are positively related is immediate from (11) and (12).

8 Conclusion

We have shown that a reversal of the prevailing assumptions on government behavior regarding sovereign debt can deliver valuable new insights. Specifically, the assumptions that governments have limited horizons and default only when unable to service their debt has made it possible to calibrate debt ratios that are markedly closer to prevailing levels than may thus far has been the case. The assumption that a government is myopic—concerned only with its current term in office—has served to introduced the concept of maximum sustainable debt, which can be viewed as a measure of Reinhart, Rogoff, and Sevastano's (2003) debt (in)tolerance. Different countries have differing debt tolerances, which depend on these countries' output volatilities (Catão and Kapur, 2004) and these countries' governments' tax raising abilities (Reinhart, Rogoff, and Sevastano, 2003). The recognition that governments' horizons may be extended beyond the current term by the prospect of reelection has provided a rationale for actual debt ratios below their maximum sustainable levels. This creates a margin of safety that allows a government if necessary to increase the debt level to the maximum sustainable ratio with little corresponding increase in the probability of default.

Our positive paper has normative implications. In particular, it implies that, for those countries that have recently experienced dramatic increases in their debt ratios yet have experienced little to no increase in their borrowing costs (France, UK, US), the present benign situation may not last, if the increase in these countries' debt ratios were to continue beyond their maximum sustainable levels. Our estimates of these levels are of course subject to considerable uncertainty, as they rely on assumptions that by necessity can be justified only partially, but the marked asymmetry in the relation between the debt ratio and the default probability before and after the maximum sustainable debt ratio does not depend on these assumptions. There is thus a point beyond which even a small increase in the debt ratio will have a large effect on the default probability, thus on the country's cost of debt.[37]

Conversely, our paper implies that, for those countries that have seen a dramatic increase in their cost of debt despite having debt ratios well short of their maximum sustainable levels (Italy, Spain), it is at least conceivable that financial markets may have overreacted. Absent any dramatic

[37] A more sanguine view may of course be suggested by the Japanese experience.

increase in these countries' growth volatility or decrease in their governments' tax raising abilities, it is not entirely clear why these countries should have experienced such large increases in borrowing costs as they did. Whilst Spain's future growth rate in all likelihood will be lower than its past rate, recall from Section 7 that the effect of the expected growth rate on the maximum sustainable debt ratio is weak in calibration. In view of government debt's bubble-like property, whereby present borrowing depends on the expectation of future borrowing, perhaps the answer lies in the coordination problems analyzed by Morris and Shin (2004), Rochet and Vives (2004), and, closer to the present analysis, Chatterjee and Eyigungor (2012).

Appendix

Derivation of (10) To obtain the equality in (10), observe that

$$\begin{aligned}
\overline{LGD} &\equiv \frac{d_M y_t - \int_0^{d_M y_t/(1+b_M)} y_{t+1} \frac{dF(y_{t+1})}{F(d_M y_t/(1+b_M))}}{d_M y_t} \\
&= 1 - \frac{\int_0^{d_M y_t/(1+b_M)} \frac{y_{t+1}}{y_t} \frac{dF(y_{t+1})}{F(d_M y_t/(1+b_M))}}{d_M} \\
&= 1 - \frac{\int_0^{d_M/(1+b_M)} g \frac{dF(g)}{F(d_M/(1+b_M))}}{d_M} \\
&= 1 - \frac{\int_0^{d_M/(1+b_M)} g \frac{d\Phi\left(\frac{\ln(g)-\mu}{\sigma}\right)}{\Phi\left(\frac{(\ln(d_M/(1+b_M))-\mu)}{\sigma}\right)}}{d_M} \\
&= 1 - \frac{\int_{-\infty}^{z_M} \exp\left(\mu + \sigma z_g\right) \frac{d\Phi(z_g)}{\Phi(z_M)}}{\exp\left(\mu + \sigma z_M\right)\left(1 + b_M\right)} \\
&= 1 - \frac{E\left[\exp\left(\sigma z_g\right) | z_g < z_M\right]}{\exp\left(\sigma z_M\right)\left(1 + b_M\right)}
\end{aligned}$$

where, as in footnote 14, $z_g \equiv \left(\ln\left(g\right) - \mu\right)/\sigma$.■

Proof of Proposition 1 We proceed in two steps. In the first step, we show that $B\left(b, z\right)$ has a unique maximum over z. In the second step, we show that $\tau\left(b\right)$ is a contraction mapping.

Step 1 Differentiate $B\left(b, z\right)$ in (7) with respect to z to obtain

$$\frac{\partial B\left(b, z\right)}{\partial z} = \frac{e^{\mu}}{1 + r}\left[e^{\sigma z}\left(1 + b\right)\left[\sigma\left[1 - \Phi\left(z\right)\right] - \phi\left(z\right)\right] + e^{\frac{\sigma^2}{2}}\phi\left(z - \sigma\right)\right] \tag{19}$$

Now use

$$e^{\frac{\sigma^2}{2}}\phi\left(z - \sigma\right) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}z^2 + \sigma z\right) = e^{\sigma z}\phi\left(z\right)$$

to conclude that $\partial B\left(b, z\right)/\partial z$ has the same sign as $\left(1 + b\right)\sigma - bL\left(z\right)$, where $L\left(z\right) \equiv \phi\left(z\right)/\left[1 - \Phi\left(z\right)\right]$ denotes the likelihood ratio of the standard normal distribution. As $L\left(z\right)$ increases monotonically from $L\left(-\infty\right) = 0$ to $L\left(+\infty\right) = +\infty$, it is clear that $\partial B\left(b, z\right)/\partial z = 0$ at the point $z = L^{-1}\left(\sigma\left(1 + 1/b\right)\right)$, which constitutes the unique maximand of $B\left(b, z\right)$.

Step 2 Consider $b_1 < b_2$. Noting from (7) that $\tau\left(b\right) \equiv \max_z B\left(b, z\right)$ is increasing in b, we have

$\tau(b_1) \leqslant \tau(b_2)$. Moreover,

$$
\begin{aligned}
\tau(b_2) &= \max\ B(b_2, z) \\
&= \frac{e^{\mu}}{1+r} \max \left[e^{\sigma z}(1+b_2)[1-\Phi(z)] + e^{\frac{\sigma^2}{2}}\Phi(z-\sigma) \right] \\
&= \frac{e^{\mu}}{1+r} \max \left[e^{\sigma z}[(1+b_1)+(b_2-b_1)][1-\Phi(z)] + e^{\frac{\sigma^2}{2}}\Phi(z-\sigma) \right] \\
&\leqslant \frac{e^{\mu}}{1+r} \max \left[e^{\sigma z}(1+b_1)[1-\Phi(z)] + e^{\frac{\sigma^2}{2}}\Phi(z-\sigma) \right] \\
&\quad + \frac{e^{\mu}}{1+r} \max\ e^{\sigma z}(b_2-b_1)[1-\Phi(z)] \\
&= \tau(b_1) + \frac{e^{\mu}}{1+r} \max\ e^{\sigma z}(b_2-b_1)[1-\Phi(z)]
\end{aligned}
$$

where the inequality is due to the general inequality $\max_z \{f(z)+g(z)\} \leqslant \max_z \{f(z)\} + \max_z \{g(z)\}$. We therefore have

$$0 \leqslant \tau(b_2) - \tau(b_1) \leqslant \frac{e^{\mu}}{1+r} \max\ e^{\sigma z}(b_2-b_1)[1-\Phi(z)]$$

Lemma 1

$$\max\ e^{\mu+\sigma z}[1-\Phi(z)] \leqslant E[g] = \exp\left(\mu + \frac{\sigma^2}{2}\right)$$

Proof of Lemma 1

$$
\begin{aligned}
e^{\mu+\sigma z}[1-\Phi(z)] &= \int_{-\infty}^{z} e^{\mu+\sigma z} d\Phi(z_g) \\
&\leqslant \int_{-\infty}^{z} e^{\mu+\sigma z_g} d\Phi(z_g) \\
&= \int_{-\infty}^{x} g dF(g) \\
&\leqslant E[g]
\end{aligned}
$$

where we have used the same transformation as in footnote (14) with $x = \exp(\mu + \sigma z)$. The result is true for all z; it is therefore true for the z that maximizes $e^{\mu+\sigma z}[1-\Phi(z)]$.■

From Lemma 1 and the assumption $E[g] < 1+r$, there exists $k < 1$ such that for all $b_1 < b_2$

$$0 \leqslant \tau(b_2) - \tau(b_1) \leqslant k(b_2 - b_1). \tag{20}$$

The function $\tau(b)$ is therefore a contraction and has a unique fixed point. We denote that point b_M.■

Proof of Proposition 2 Differentiate $b_M = \tau(b_M)$ with respect to $\Xi \in \{\mu, \sigma, r\}$ to obtain

$$\frac{\partial b_M}{\partial \Xi} = \tau'(b_M)\frac{\partial b_M}{\partial \Xi} + \frac{\partial \tau(b_M)}{\partial \Xi}$$

$$\Leftrightarrow \left[1 - \tau'(b_M)\right]\frac{\partial b_M}{\partial \Xi} = \frac{\partial \tau(b_M)}{\partial \Xi}$$

Use $\tau'(b_M) < 1$ ($\tau(b)$ being a contraction mapping) to conclude that $sign\{\partial b_M/\partial \Xi\} = sign\{\partial \tau(b_M)/\partial \Xi\}$. From (7) and $\tau(b) \equiv \max_z B(b,z)$, it is immediate that $\partial \tau(b)/\partial \mu > 0$ and $\partial \tau(b)/\partial r < 0$; it is therefore the case that $\partial b_M/\partial \mu > 0$ and $\partial b_M/\partial r < 0$. From (7) and footnote (14), we have

$$\frac{\partial B(b,z)}{\partial \sigma} = \frac{e^{\mu}}{1+r}\left[ze^{\sigma z}(1+b)\left[1-\Phi(z)\right] + \int_{-\infty}^{z} z_g \exp(\sigma z_g)\, d\Phi(z_g)\right]$$

A sufficient condition for $\partial B(b,z)/\partial \sigma < 0$ is $z < 0$. At $b = b_M$ and $z = z_M$, $z_M < 0$ amounts to the reasonable assumption that countries have probability of default $\Phi(z_M) < 1/2$ at their maximum sustainable debt ratio. The intuitive result $\partial b_M/\partial \sigma < 0$ follows.

The FOC for z_M is

$$\left.\frac{\partial B(b_M,z)}{\partial z}\right|_{z=z_M} = \sigma(1+b_M)\left[1-\Phi(z_M)\right] - b_M\phi(z_M) = 0 \tag{21}$$

It can be rewritten as

$$L(z_M) = \sigma\left(1 + \frac{1}{b_M}\right) \tag{22}$$

where $L(z) \equiv \phi(z)/\left[1-\Phi(z)\right]$ denotes the hazard rate; $L'(z) > 0$ as the normal distribution has monotone hazard rate.[38] Differentiate (22) with respect to μ, σ, and r to obtain

$$L'(z_M)\frac{\partial z_M}{\partial \mu} = -\frac{\sigma}{b_M^2}\frac{\partial b_M}{\partial \mu} < 0 \tag{23}$$

$$L'(z_M)\frac{\partial z_M}{\partial \sigma} = \frac{L(z_M)}{\sigma} - \frac{\sigma}{b_M^2}\frac{\partial b_M}{\partial \sigma} > 0 \tag{24}$$

[38] A cdf $F(x)$ with associated pdf $f(x)$ has monotone hazard rate when $f(x)/\left[1-F(x)\right]$ increases in x.

and

$$L'(z_M)\frac{\partial z_M}{\partial r} = -\frac{\sigma}{b_M^2}\frac{\partial b_M}{\partial r} > 0\blacksquare \tag{25}$$

Proof of Proposition 3 Consider debt dy_t due in period $t+1$. A government that engages only in excusable default exhausts all $t+1$ borrowing possibilities, specifically $b_M y_{t+1}$, before declaring default. Default therefore occurs with probability

$$\begin{aligned}
\Pr[y_{t+1} + b_M y_{t+1} < dy_t] &= \Pr\left[g < \frac{d}{1+b_M}\right] \\
&= \Phi\left(\frac{\ln(d) - \ln(1+b_M) - \mu}{\sigma}\right) \\
&\equiv \Phi(z)
\end{aligned}$$

where z is defined analogously to (3)

$$z \equiv \frac{\ln(d) - \ln(1+b_M) - \mu}{\sigma}$$

Use $\partial b_M/\partial\mu > 0$, $\partial b_M/\partial\sigma < 0$ for $z < 0$, and $\partial b_M/\partial r < 0$, to conclude that $\partial z/\partial\mu < 0$, $\partial z/\partial\sigma > 0$ for $z < 0$, and $\partial z/\partial r > 0$, respectively.$\blacksquare$

Proof of Proposition 4 Use (8) and (23) to write

$$\begin{aligned}
\frac{\partial \ln(d_M)}{\partial\mu} &= \frac{\frac{\partial b_M}{\partial\mu}}{1+b_M} + 1 + \sigma\frac{\partial z_M}{\partial\mu} \\
&= \frac{\partial b_M}{\partial\mu}\left[\frac{1}{1+b_M} - \frac{\sigma^2}{L'(z_M)\,b_M^2}\right] + 1
\end{aligned} \tag{26}$$

Now use the definition of $L(z)$ and (22) to write

$$\begin{aligned}
L'(z_M) &= -z_M L(z_M) + L^2(z_M) \\
&= \sigma\frac{1+b_M}{b_M}\left[\sigma\frac{1+b_M}{b_M} - z_M\right]
\end{aligned}$$

$$\Leftrightarrow \frac{L'(z_M)\,b_M^2}{\sigma^2} = (1+b_M)\left[1 + b_M - \frac{z_M b_M}{\sigma}\right]$$

The assumption $z_M < 0$ implies that $L'(z_M)\,b_M^2/\sigma^2 > 1 + b_M$, in turn implying that the term in square brackets on the RHS of (26) is positive; $\partial \ln(d_M)/\partial\mu > 0$ is then an immediate consequence of $\partial b_M/\partial\mu > 0$ established in Proposition **2**; $\partial d_M/\partial\mu > 0$ follows.

Use (8) and (25) to write

$$\begin{aligned}\frac{\partial\ln(d_M)}{\partial r} &= \frac{\frac{\partial b_M}{\partial r}}{1+b_M} + \sigma\frac{\partial z_M}{\partial r} \\ &= \frac{\partial b_M}{\partial r}\left[\frac{1}{1+b_M} - \frac{\sigma^2}{L'(z_M)\,b_M^2}\right] \end{aligned} \tag{27}$$

The term in square brackets on the RHS of (27) is identical to that in (26), which we have just shown to be positive; $\partial\ln(d_M)/\partial r < 0$ is then an immediate consequence of $\partial b_M/\partial r < 0$ established in Proposition **2**; $\partial d_M/\partial r < 0$ follows.

Finally, use (9) to write

$$\frac{\partial\ln(R_M)}{\partial r} = \sigma\frac{\partial z_M}{\partial r} + \frac{L'(z_M)}{L(z_M)}\frac{\partial z_M}{\partial r} > 0$$

where we have used $\partial z_M/\partial r > 0$ from Proposition **2** and $L'(z) > 0$; $\partial R_M/\partial r > 0$ follows.■

Proof of Proposition 5 Differentiate (18) with respect to d and d', recall that d^* denotes the optimal d, and use the Envelope Theorem to write

$$u'\left(1+B(d^*)-d'\right)B'(d^*) + \frac{\theta}{1+r}E\left[g^{-a}v'\left(\frac{d^*}{g}\right)\right] = 0 \tag{28}$$

and

$$v'\left(d'\right) = -u'\left(1+B(d^*)-d'\right) < 0 \tag{29}$$

Note that (28) and (29) together imply $B'(d^*) > 0$. Now use the Implicit Function Theorem to conclude

$$sign\left\{\frac{\partial d^*}{\partial\theta}\right\} = sign\left\{\frac{1}{1+r}E\left[g^{-a}v'\left(\frac{d^*}{g}\right)\right]\right\} = -1$$

The remaining results follow from $B'(d^*) > 0$ and $F'(.) > 0$.■

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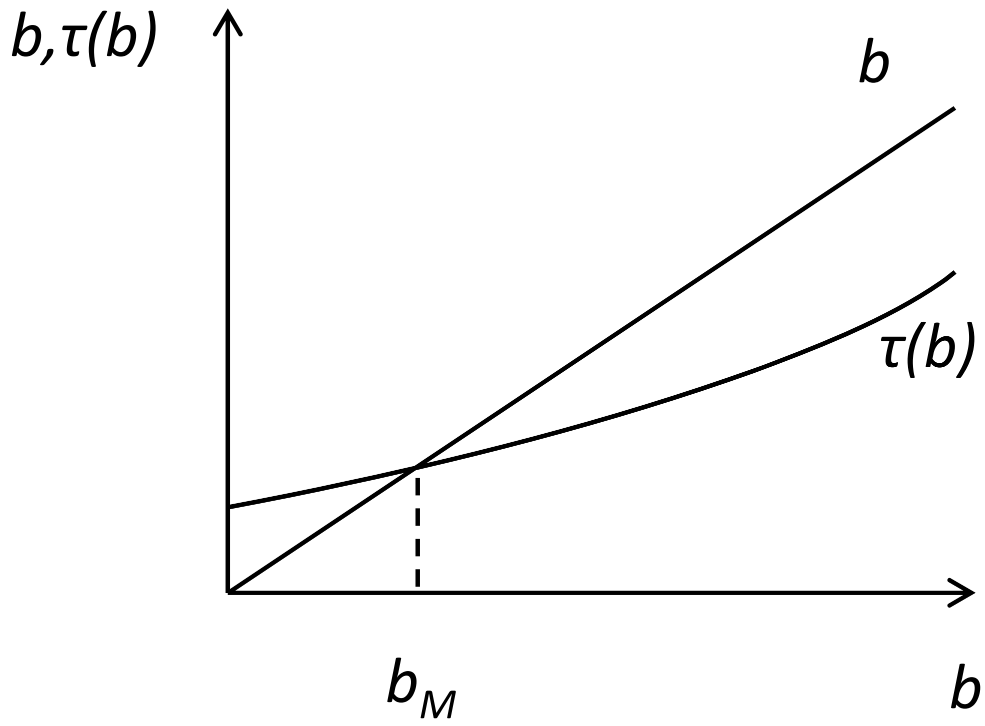
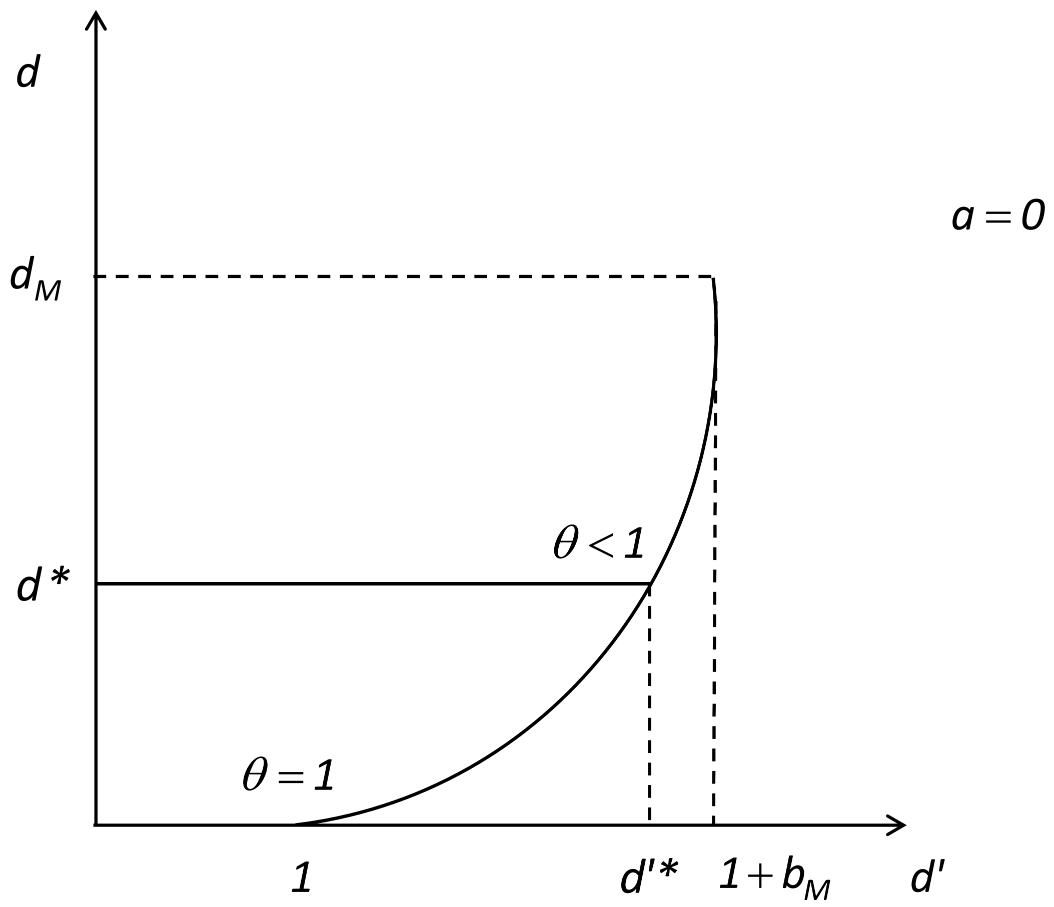


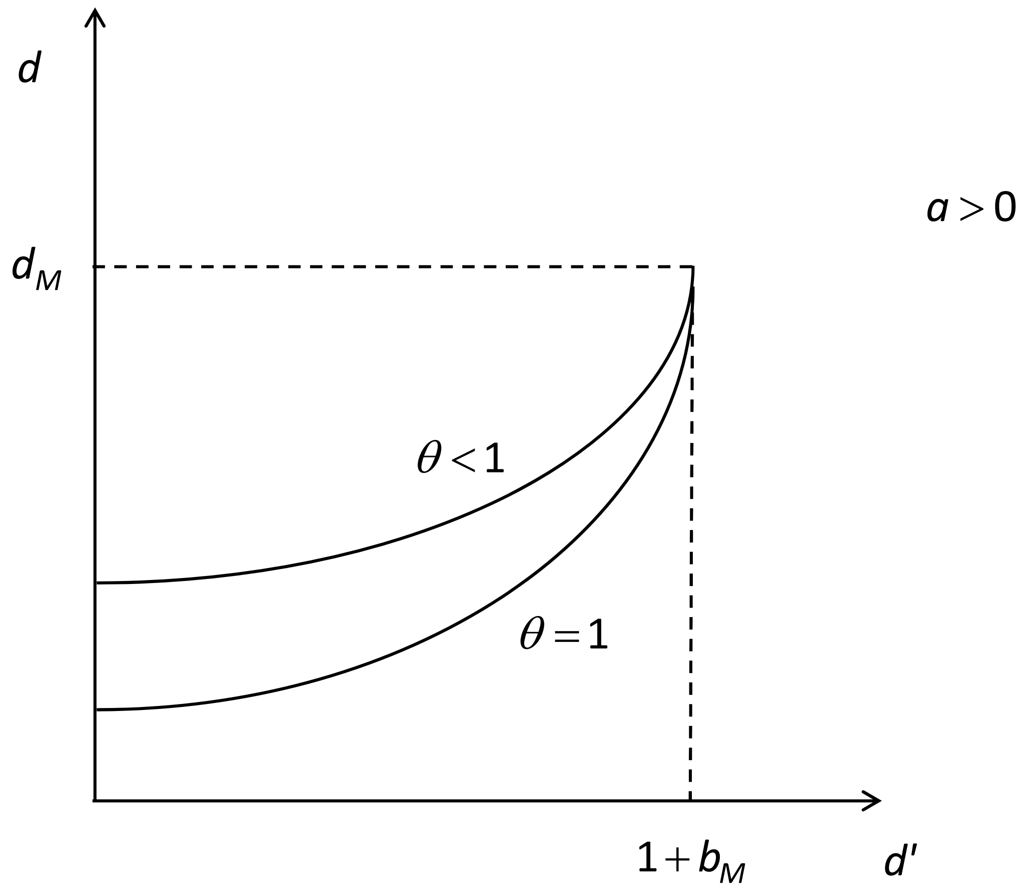
Maximum Sustainable Borrowing b_M

Figure 1



Debt Dynamics under Risk Neutrality

Figure 2



Debt Dynamics under Risk Aversion

Figure 3

Table 1 (%)

COUNTRY	μ	σ	Gvt.Inc./GDP	d_m	PD_m	D, 80-11	PD, 80-11	D, 11	PD, 11	d_M	PD_M
Argentina	2.5	6.0	28	7	0	73	82	45	0	61	3.09
Australia	3.1	1.6	33	21	0	20	0	24	0	166	0.62
Austria	2.1	1.5	49	44	0	64	0	72	0	224	0.59
Belgium	1.9	1.6	48	33	0	109	0	97	0	213	0.59
Brazil	2.7	3.3	34	16	0	69	0	65	0	111	1.46
Canada	2.5	2.1	42	24	0	77	0	85	0	168	0.86
Chile	4.5	4.7	23	10	0	10	0	11	0	71	2.18
Czech Republic	2.8	2.9	39	35	0	25	0	40	0	138	1.26
Denmark	1.6	2.2	58	36	0	51	0	44	0	216	0.88
Ecuador	3.1	3.4	24	12	0	34	0	18	0	79	1.50
Egypt	4.6	2.0	26	45	0	86	0	76	0	141	0.77
Finland	2.4	3.3	53	25	0	34	0	49	0	169	1.45
France	1.8	1.4	49	61	0	50	0	86	0	222	0.54
Germany	1.7	2.0	45	24	0	61	0	80	0	173	0.82
Greece	1.4	2.9	34	15	0	85	0	165	100	107	1.29
Hungary	1.1	3.6	45	19	0	65	0	80	0	122	1.65
Iceland	2.6	3.5	42	26	0	45	0	99	0	128	1.59
Indonesia	5.0	3.9	18	16	0	49	0	24	0	62	1.75
Israel	4.1	2.3	44	32	0	86	0	74	0	202	0.95
Italy	1.3	1.8	44	24	0	108	0	120	0	172	0.73
Japan	2.1	2.5	30	20	0	118	96	229	100	106	1.06
Malaysia	5.7	3.8	26	24	0	47	0	53	0	103	1.66
Mexico	2.6	3.7	20	9	0	43	0	44	0	61	1.65
Netherlands	2.2	1.8	46	36	0	59	0	65	0	194	0.71
New Zealand	2.2	2.1	34	19	0	42	0	38	0	135	0.85

Norway	2.6	1.8	53	36	0	42	0	50	0	236	0.70
Peru	3.1	6.1	19	7	0	35	0	21	0	44	3.13
Romania	1.1	5.5	31	7	0	23	0	33	0	67	2.77
Russia	1.7	6.8	37	7	0	29	0	12	0	72	3.69
South Africa	2.4	2.4	27	17	0	35	0	39	0	101	1.00
Spain	2.5	2.1	37	27	0	47	0	69	0	150	0.84
Sweden	2.3	2.2	56	52	0	55	0	38	0	216	0.91
Switzerland	1.8	1.7	34	26	0	50	0	47	0	141	0.67
Turkey	4.1	4.4	32	14	0	53	0	39	0	97	2.04
Ukraine	-0.8	10.6	39	7	0	32	0	36	0	53	6.81
United Kingdom	2.3	2.2	38	21	0	46	0	82	0	149	0.89
United States	2.6	2.0	32	25	0	65	0	103	0	135	0.81
Uruguay	2.5	4.6	30	14	0	73	0	55	0	80	2.18
Venezuela	1.7	6.8	31	6	0	42	0	47	0	61	3.64
Vietnam	6.3	2.4	25	23	0	43	0	50	0	152	0.95